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The Thurston norm, fibered manifolds and twisted Alexander polynomials

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Abstract

Every element in the first cohomology group of a 3-manifold is dual to embedded surfaces. The Thurston norm measures the minimal 'complexity' of such surfaces. For instance the Thurston norm of a knot complement determines the genus of the knot in the 3-sphere. We show that the degrees of twisted Alexander polynomials give lower bounds on the Thurston norm, generalizing work of McMullen and Turaev. Our bounds attain their most concise form when interpreted as the degrees of the Reidemeister torsion of a certain twisted chain complex. We show that these lower bounds give the correct genus bounds for all knots with 12 crossings or less, including the Conway knot and the Kinoshita–Terasaka knot which have trivial Alexander polynomial.

We also give obstructions to fibering 3-manifolds using twisted Alexander polynomials and detect all knots with 12 crossings or less that are not fibered. For some of these it was unknown whether or not they are fibered. Our work in particular extends the fibering obstructions of Cha to the case of closed manifolds.
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1. Introduction

1.1. Definitions and history

Let M be a 3-manifold. Throughout the paper we will assume that all 3-manifolds are compact, orientable and connected. Let $\phi \in H^1(M)$ (integral coefficients are understood). The *Thurston norm* of

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ϕ is defined as

$$\|\phi\|_T = \min\{\chi_-(S) \mid S \subset M \text{ properly embedded surface dual to } \phi\}.$$

Here, given a surface S with connected components $S_1 \cup \cdots \cup S_k$, we define $\chi_-(S) = \sum_{i=1}^k \max\{-\chi(S_i), 0\}$. Thurston [25] showed that this defines a seminorm on $H^1(M)$ which can be extended to a seminorm on $H^1(M; \mathbb{R})$. As an example consider $X(K) := S^3 \setminus \nu K$, where $K \subset S^3$ is a knot and νK denotes an open tubular neighborhood of K in S^3. Let $\phi \in H^1(X(K))$ be a generator, then it is easy to see that $\|\phi\|_T = 2 \operatorname{genus}(K) - 1$.

It is a classical result of Alexander that

$$2\operatorname{genus}(K) \geq \deg(\Delta_K(t)),$$

where $\Delta_K(t)$ denotes the Alexander polynomial of a knot K. In recent years this was greatly generalized. Let M be a 3-manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M) \cong \operatorname{Hom}(H_1(M), \mathbb{Z})$ be primitive, i.e., the corresponding homomorphism $\phi : H_1(M) \to \mathbb{Z}$ is surjective. Then McMullen [20] showed that if the Alexander polynomial $\Delta_1(t) \in \mathbb{Q}[t^{\pm 1}]$ of (M, ϕ) is non-zero, then

$$\|\phi\|_T \geq \deg(\Delta_1(t)) - (1 + b_3(M)).$$

Here $b_3(M)$ denotes the third Betti number of M, in particular $b_3(M) = 1$ if M is closed and $b_3(M) = 0$ if M has boundary. An alternative proof for closed manifolds was given by Vidussi [29,30] using results of Kronheimer and Mrowka [18] and Meng and Taubes [21] in Seiberg–Witten theory.

Harvey [11] in the general case and Cochran [3] in the knot complement case generalized McMullen's inequality. They showed that the degrees of *higher-order* Alexander polynomials which are defined over non-commutative polynomial rings give lower bounds on the Thurston norm. Later Harvey's work [11] was refined by Turaev [28].

In this paper we will show how the degrees of *twisted Alexander polynomials* give lower bounds on the Thurston norm.

1.2. Twisted Alexander polynomials and Reidemeister torsion

In the following let $\mathbb{F}$ be field. Let $\phi \in H^1(M) \cong \operatorname{Hom}(\pi_1(M), \mathbb{Z})$ and $\alpha : \pi_1(M) \to \operatorname{GL}(\mathbb{F}, k)$ a representation. Then $\alpha \otimes \phi$ induces an action of $\pi_1(M)$ on $\mathbb{F}^k \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] =: \mathbb{F}^k[t^{\pm 1}]$ and we can therefore consider the twisted homology $\mathbb{F}[t^{\pm 1}]$-module $H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}])$. We define $\Delta_i^\alpha(t) \in \mathbb{F}[t^{\pm 1}]$ to be its order; it is called *the ith twisted Alexander polynomial of (M, ϕ, α)* and well-defined up to multiplication by a unit in $\mathbb{F}[t^{\pm 1}]$. The twisted Alexander polynomial of a knot was introduced by Lin [19] in 1990. In this paper we use the above homological definition of Kirk and Livingston [16]. These polynomials can be computed efficiently using Fox calculus and Poincaré duality for twisted homology. We refer to Section 2 for more details.

If ∂M is empty or consists of tori and if $\Delta_1^\alpha(t) \neq 0$, then we will show that $H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}] \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t)) = 0$ for all i. Therefore the Reidemeister torsion $\tau(M, \phi, \alpha) \in \mathbb{F}(t)$ is defined (cf. [26] for a definition) and (cf. [26, p. 20] or [16, Theorem 3.4])

$$\tau(M, \phi, \alpha) = \prod_{i=0}^2 \Delta_i^\alpha(t)^{(-1)^{i+1}} \in \mathbb{F}(t).$$

The equality holds up to multiplication by a unit in $\mathbb{F}[t^{\pm 1}]$. For $f(t)/g(t) \in \mathbb{F}(t)$ we define $\deg(f(t)/g(t)) := \deg(f(t)) - \deg(g(t))$ for $f(t), g(t) \in \mathbb{F}[t^{\pm 1}]$. This allows us to consider $\deg(\tau(M, \phi, \alpha))$.

1.3. Lower bounds on the Thurston norm

The following theorem shows that the degrees of twisted Alexander polynomials can be used to give lower bounds on the Thurston norm.

Theorem 1.1. *Let M be a 3-manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M)$ be non-trivial and $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ a representation such that $\Delta_1^{\alpha}(t) \neq 0$. Then*

$$\|\phi\|_T \geq \frac{1}{k} \deg(\tau(M, \phi, \alpha)).$$

Equivalently,

$$\|\phi\|_T \geq \frac{1}{k} \left(\deg(\Delta_1^{\alpha}(t)) - \deg(\Delta_0^{\alpha}(t)) - \deg(\Delta_2^{\alpha}(t)) \right).$$

The proof of Theorem 1.1 is partly based on ideas of McMullen [20] and Turaev [28]. For one-dimensional representations it is easy to determine $\Delta_0^{\alpha}(t)$ and $\Delta_2^{\alpha}(t)$ and one can easily show that Theorem 1.1 contains McMullen's bound for one-variable Alexander polynomials ([20, Proposition 6.1]) and results of Turaev [27].

In Theorem 4.1 we give lower bounds in the case $\Delta_1^{\alpha}(t) = 0$ for certain ϕ's. In [7] we introduce twisted Alexander norms (similar to McMullen's Alexander norm [20]) which are well-suited to study the Thurston norm of link complements. We also refer to [6] for a further extension of Theorem 1.1.

1.4. Fibered manifolds

Let $\phi \in H^1(M)$ be non-trivial. We say (M, ϕ) *fibers over* S^1 if the homotopy class of maps $M \to S^1$ induced by $\phi : \pi_1(M) \to H_1(M) \to \mathbb{Z}$ contains a representative that is a fiber bundle over S^1. If K is a fibered knot, i.e., if $X(K)$ fibers, then it is a classical result of Neuwirth that K satisfies

$$\Delta_K(t) \text{ is monic} \quad \text{and} \quad \deg(\Delta_K(t)) = 2\,\mathrm{genus}(K). \tag{1}$$

Theorem 1.2. *Assume that (M, ϕ) fibers over S^1 and that $M \neq S^1 \times D^2, M \neq S^1 \times S^2$. Let $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ be a representation. Then $\Delta_1^{\alpha}(t) \neq 0$ and*

$$\|\phi\|_T = \frac{1}{k} \deg(\tau(M, \phi, \alpha)).$$

This result clearly generalizes the first classical condition on fibered knots. McMullen, Cochran, Harvey and Turaev prove corresponding theorems in their respective papers [20,3,11,28].

Now let R be a Noetherian unique factorization domain (henceforth UFD), for example $R = \mathbb{Z}$. Given a representation $\pi_1(M) \to \mathrm{GL}(R, k)$ Cha [1] defined a twisted Alexander polynomial $\Delta_1^{\alpha}(t) \in R[t^{\pm 1}]$, which is well-defined up to multiplication by a unit in $R[t^{\pm 1}]$. Cha showed that for a fibered knot the polynomials $\Delta_1^{\alpha}(t)$ are monic [1]. (Recall that a polynomial is called monic, if its highest and lowest coefficient are units in R.) Using Theorem 1.2 we obtain the following theorem.

Theorem 1.3. *Let M be a 3-manifold. Let $\phi \in H^1(M)$ be non-trivial such that (M, ϕ) fibers over S^1 and such that $M \neq S^1 \times D^2$, $M \neq S^1 \times S^2$. Let R be a Noetherian UFD and let $\alpha : \pi_1(M) \to \mathrm{GL}(R, k)$ be a representation. Then $\Delta_1^\alpha(t) \in R[t^{\pm 1}]$ is monic and*

$$\|\phi\|_T = \frac{1}{k} \deg(\tau(M, \phi, \alpha)).$$

In fact in Proposition 6.1 we show that if the fibering obstruction of Theorem 1.2 vanishes, then the conclusion of Theorem 1.3 holds. This shows that the obstructions of Theorem 1.2 contain Neuwirth's and Cha's [1] obstructions for fibered knots and extend them to closed 3-manifolds. Theorem 1.2 is also closely related to the result of [10] on fibered knots.

1.5. Examples

We give two main examples in Section 5. First we show that the lower bounds of Theorem 1.1 give, for appropriate representations, the correct genus bounds for all knots with up to 12 crossings. These genera have been found by Gabai, Rasmussen, Stoimenow et al. (cf. [2] and [23]). Note that some knots with up to 12 crossings have trivial Alexander polynomial, and hence the genus bounds of McMullen, Cochran and Harvey vanish. These, and all later computations in this paper, were done using the program *KnotTwister* [5].

Second we apply Theorem 1.2 to study the fiberedness of knots. It is known that a knot K with 11 or fewer crossings is fibered if and only if K satisfies Neuwirth's condition (1). Hirasawa and Stoimenow [23] had started a program to find all fibered 12-crossing knots. Using methods of Gabai they showed that except for thirteen knots a 12-crossing knot is fibered if and only if it satisfies condition (1). Furthermore they showed that among these 13 knots the knots 12_{1498}, 12_{1502}, 12_{1546} and 12_{1752} are not fibered even though they satisfy condition (1). Using Theorem 1.2 we showed the non-fiberedness of these 4 knots and we also showed that the remaining 9 knots are not fibered either. These 9 knots are:

$$12_{1345}, 12_{1567}, 12_{1670}, 12_{1682}, 12_{1771}, 12_{1823}, 12_{1938}, 12_{2089}, 12_{2103}.$$

This result completes the classification of all fibered 12-crossing knots. We note that later Jacob Rasmussen also showed that these 13 knots are not fibered using knot Floer homology (cf. [22, Section 3]).

1.6. Outline of the paper

In Section 2 we give a definition of twisted Alexander polynomials and we discuss the Alexander polynomials of 3-manifolds. We give the proofs of Theorems 1.1 and 1.2 in Section 3. In Section 4 we discuss the case that $\Delta_1^\alpha(t) = 0$. We discuss the examples in Section 5. In Section 6 we prove Theorem 1.3.

Notations and conventions: We assume that all 3-manifolds are compact, oriented and connected. All homology groups and all cohomology groups are with respect to $\mathbb{Z}$-coefficients, unless it specifically says otherwise. For a link L in S^3, $X(L)$ denotes the exterior of L in S^3. (That is, $X(L) = S^3 \setminus \nu L$ where νL is an open tubular neighborhood of L in S^3.) $\mathbb{F}$ will always denote a field. We identify the group ring $\mathbb{F}[\mathbb{Z}]$ with $\mathbb{F}[t^{\pm 1}]$. For a 3-manifold M we use the canonical isomorphisms to identify $H^1(M) = \mathrm{Hom}(H_1(M), \mathbb{Z}) = \mathrm{Hom}(\pi_1(M), \mathbb{Z})$. Hence sometimes $\phi \in H^1(M)$ is regarded as a homomorphism $\phi : \pi_1(M) \to \mathbb{Z}$ (or $\phi : H_1(M) \to \mathbb{Z}$) depending on the context.

2. The twisted Alexander polynomials and duality

2.1. Twisted homology groups

We first give a definition of twisted homology groups and discuss some of their properties. Let X be a topological space, $Y \subset X$ a (possibly empty) subset and $x_0 \in X$ a point. Let R be a ring (e.g. $R = \mathbb{F}$ or $R = \mathbb{F}[t^{\pm 1}]$) and $\beta : \pi_1(X, x_0) \to \mathrm{GL}(R, k)$ a representation. This naturally induces a left $\mathbb{Z}[\pi_1(X, x_0)]$-module structure on R^k.

Denote by $\tilde{X}$ the set of all homotopy classes of paths starting at x_0 with the usual topology. Then the evaluation map $p : \tilde{X} \to X$ is the universal cover of X. Note that $g \in \pi_1(X, x_0)$ naturally acts on $\tilde{X}$ on the right by precomposing any path by g^{-1}.

Given $Y \subset X$ we let $\tilde{Y} = p^{-1}(Y) \subset \tilde{X}$. Then the above $\pi_1(X, x_0)$ action on $\tilde{X}$ gives rise to a right $\mathbb{Z}[\pi_1(X, x_0)]$-module structure on the chain groups $C_*(\tilde{X})$, $C_*(\tilde{Y})$ and $C_*(\tilde{X}, \tilde{Y})$. Therefore we can form the tensor product over $\mathbb{Z}[\pi_1(X, x_0)]$ with R^k, we define

$$H_i^{\beta}(X; R^k) = H_i(C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} R^k),$$
$$H_i^{\beta}(Y \subset X; R^k) = H_i(C_*(\tilde{Y}) \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} R^k),$$
$$H_i^{\beta}(X, Y; R^k) = H_i(C_*(\tilde{X}, \tilde{Y}) \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} R^k).$$

Sometimes we refer to $H_i^{\beta}(Y \subset X; R^k)$ as *twisted subspace homology*. Note that if we have inclusions $Z \subset Y \subset X$ then we get an induced map $H_i^{\beta}(Z \subset X; R^k) \to H_i^{\beta}(Y \subset X; R^k)$. Also note that we have an exact sequence of complexes

$$0 \to C_i(\tilde{Y}) \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} R^k \to C_i(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} R^k \to C_i(\tilde{X}, \tilde{Y}) \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} R^k \to 0$$

which gives rise to a long exact sequence

$$\cdots \to H_i^{\beta}(Y \subset X; R^k) \to H_i^{\beta}(X; R^k) \to H_i^{\beta}(X, Y; R^k) \to \cdots . \tag{2}$$

Now denote by $Y_i, i \in I$, the path connected components of Y. Pick base points $y_i \in Y_i, i \in I$, and paths $\gamma_i : [0, 1] \to Y$ with $\gamma_i(0) = y_i$ and $\gamma_i(1) = x_0$. Then we can get induced representations $\beta_i(\gamma_i) : \pi_1(Y_i, y_i) \to \pi_1(X, y_i) \to \pi_1(X, x_0) \to \mathrm{GL}(R, k)$ and induced homology groups $H_j^{\beta_i(\gamma_i)}(Y_i; R^k)$ using the universal cover of Y_i.

Lemma 2.1. *Given γ_i there exists a canonical isomorphism*

$$H_j^{\beta_i}(Y_i \subset X; R^k) \cong H_j^{\beta_i(\gamma_i)}(Y_i; R^k).$$

Proof. Let K be the image of $\pi_1(Y_i, y_i)$ under the map $\iota(\gamma_i) : \pi_1(Y_i, y_i) \to \pi_1(X, y_i) \to \pi_1(X, x_0)$ induced by γ_i. Denote by $\widetilde{Y_i}^K$ the cover of Y_i corresponding to $\pi_1(Y_i, y_i) \to K$. More precisely, we take

$$\widetilde{Y_i}^K = \{\sigma : [0, 1] \to Y_i \mid \sigma(0) = y_i\}/ \sim$$

where $\sim$ is the equivalence relation given by

$$\sigma_1 \sim \sigma_2 \quad \text{if } \sigma_1(1) = \sigma_2(1) \quad \text{and} \quad \iota(\gamma_i)(\sigma_1 \sigma_2^{-1}) = e \in K.$$

Then we get a well-defined injective map

$$\widetilde{Y}_i{}^K \to \widetilde{Y}_i \subset \widetilde{X}$$
$$[\sigma] \mapsto [\gamma_i^{-1}\sigma].$$

We will use this injection to identify $\widetilde{Y}_i{}^K$ with its image in $\widetilde{Y}_i$. Note that $\widetilde{Y}_i$ is the disjoint union of copies of $\widetilde{Y}_i{}^K$ indexed by $\pi_1(X, x_0)/K$. In particular a singular simplex in $\widetilde{Y}_i$ is of the form σg for a singular simplex σ in $\widetilde{Y}_i{}^K$ and an element $g \in \pi_1(X, x_0)$. Mapping

$$\sigma g \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} v \mapsto \sigma \otimes_{\mathbb{Z}[K]} \beta(g)v$$

(with $v \in R^k$) induces an isomorphism $C_*(\widetilde{Y}_i) \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} R^k \cong C_*(\widetilde{Y}_i{}^K) \otimes_{\mathbb{Z}[K]} R^k$ of chain complexes. Denote the universal cover of Y_i by $\widetilde{Y}_i{}^{\pi_1(Y_i, y_i)}$. It is easy to see that the projection induced map $C_*(\widetilde{Y}_i{}^{\pi_1(Y_i, y_i)}) \to C_*(\widetilde{Y}_i{}^K)$ gives rise to an isomorphism of chain complexes:

$$C_*(\widetilde{Y}_i{}^K) \otimes_{\mathbb{Z}[K]} R^k \cong C_*(\widetilde{Y}_i{}^{\pi_1(Y_i, y_i)}) \otimes_{\mathbb{Z}[\pi_1(Y_i, y_i)]} R^k. \quad \square$$

Note that the isomorphism of the lemma only depends on the choice of γ_i, and we call the isomorphism the *isomorphism induced by γ_i*.

It is clear that the isomorphism type of $H_j^{\beta}(X, Y; R^k)$ does not depend on the choice of the base point. In most situations we can and will therefore suppress the base point in the notation and the arguments. We will also normally write β instead of $\beta(\gamma_i)$. Furthermore we write $H_j^{\beta}(Y; R^k) = \oplus_{i \in I} H_j^{\beta}(Y_i; R^k)$. With these conventions the long exact sequence (2) induces a long exact sequence

$$\cdots \to H_j^{\beta}(Y; R^k) \to H_j^{\beta}(X; R^k) \to H_j^{\beta}(X, Y; R^k) \to \cdots.$$

The isomorphism type of the sequence is independent of all the choices made.

2.2. The twisted Alexander polynomials

For the remainder of this section we assume that M is a 3-manifold and $\phi \in H^1(M)$. Let $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ be a representation. We can now define a left $\mathbb{Z}[\pi_1(M)]$-module structure on $\mathbb{F}^k \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] =: \mathbb{F}^k[t^{\pm 1}]$ via $\alpha \otimes \phi$ as follows:

$$g \cdot (v \otimes p) := (\alpha(g) \cdot v) \otimes (\phi(g) \cdot p) = (\alpha(g) \cdot v) \otimes (t^{\phi(g)} p)$$

where $g \in \pi_1(M), v \otimes p \in \mathbb{F}^k \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] = \mathbb{F}^k[t^{\pm 1}]$. Put differently, we get a representation $\alpha \otimes \phi : \pi_1(M) \to \mathrm{GL}(\mathbb{F}[t^{\pm 1}], k)$. We call $H_i^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}])$ the ith *twisted Alexander module of* (M, ϕ, α). Usually we drop the notation ϕ and write $H_*^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}])$. Note that $H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}])$ is a finitely generated module over the PID $\mathbb{F}[t^{\pm 1}]$. Therefore there exists an isomorphism

$$H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \cong \mathbb{F}[t^{\pm 1}]^f \oplus \bigoplus_{i=1}^{l} \mathbb{F}[t^{\pm 1}]/(p_i(t))$$

for $p_1(t), \ldots, p_l(t) \in \mathbb{F}[t^{\pm 1}] \setminus \{0\}$. We define

$$
\Delta^\alpha_{M,\phi,i}(t) := \begin{cases} \prod_{i=1}^l p_i(t), & \text{if } f = 0 \\ 0, & \text{if } f > 0. \end{cases}
$$

This is called the ith *twisted Alexander polynomial* of (M, ϕ, α). We furthermore define $\tilde{\Delta}^\alpha_{M,\phi,i}(t) :=$ $\prod_{i=1}^k p_i(t)$ regardless of f. In most cases we drop the notations M and ϕ and write $\Delta^\alpha_i(t)$ and $\tilde{\Delta}^\alpha_i(t)$. It follows from the structure theorem of finitely generated modules over a PID that these polynomials are well-defined up to multiplication by a unit in $\mathbb{F}[t^{\pm 1}]$.

For an oriented knot K we always assume that ϕ denotes the generator of $H^1(X(K))$ given by the orientation. If $\alpha : \pi_1(X(K)) \to \mathrm{GL}(\mathbb{Q}, 1)$ is the trivial representation then the Alexander polynomial $\Delta^\alpha_1(t)$ equals the classical Alexander polynomial $\Delta_K(t) \in \mathbb{Q}[t^{\pm 1}]$ of the knot K.

Let $f = \sum_{i=m}^n a_i t^i \in \mathbb{F}[t^{\pm 1}] \setminus \{0\}$ with $a_m \neq 0, a_n \neq 0$. Then we define $\deg(f) = n - m$. The following observation follows immediately from the classification theorem of finitely generated modules over a PID.

Lemma 2.2. $H^\alpha_i(M; \mathbb{F}^k[t^{\pm 1}])$ *is a finite-dimensional $\mathbb{F}$-vector space if and only if $\Delta^\alpha_i(t) \neq 0$. If $\Delta^\alpha_i(t) \neq 0$, then*

$$
\deg(\Delta^\alpha_i(t)) = \dim_\mathbb{F}(H^\alpha_i(M; \mathbb{F}^k[t^{\pm 1}])).
$$

Furthermore $\deg(\tilde{\Delta}^\alpha_i(t)) = \dim_\mathbb{F}(\mathrm{Tor}_{\mathbb{F}[t^{\pm 1}]}(H^\alpha_i(M; \mathbb{F}^k[t^{\pm 1}])))$.

2.3. Duality for twisted homology

In this section we discuss a duality theorem for twisted homology which we will need to compute higher twisted Alexander polynomials of 3-manifolds and which will also play an important role in the proof of Proposition 3.6.

Let $\mathbb{F}$ be a field with (possibly trivial) involution $f \mapsto \overline{f}$. We equip $\mathbb{F}^k$ with the standard hermitian inner product $\langle v, w \rangle = v^t \overline{w}$ (where we view elements in $\mathbb{F}^k$ as column vectors). We extend the involution on $\mathbb{F}$ to $\mathbb{F}[t^{\pm 1}]$ by taking $t \mapsto t^{-1}$. We equip $\mathbb{F}^k[t^{\pm 1}]$ with the hermitian inner product defined by $\langle vt^i, wt^j \rangle := \langle v, w \rangle t^i t^{-j}$ for all $v, w \in \mathbb{F}^k$.

In the following let $R = \mathbb{F}$ or $R = \mathbb{F}[t^{\pm 1}]$. Let $\beta : \pi_1(M) \to \mathrm{GL}(R, k)$ be a representation. There exists a unique representation $\overline{\beta} : \pi_1(M) \to \mathrm{GL}(R, k)$ such that

$$
\langle \beta(g^{-1})v, w \rangle = \langle v, \overline{\beta}(g)w \rangle
$$

for all $v, w \in \mathbb{F}^k, g \in \pi_1(M)$.

The following lemma is a variation on [16, p. 639].

Lemma 2.3. *Let X be an n-manifold and $\beta : \pi_1(X) \to \mathrm{GL}(R, k)$ a representation. Then*

$$
H^\beta_{n-i}(X; R^k) \cong \mathrm{Hom}_R(H^{\overline{\beta}}_i(X, \partial X; R^k), R) \oplus \mathrm{Ext}_R(H^{\overline{\beta}}_{i-1}(X, \partial X; R^k), R)
$$

as R-modules.

Proof. Let $\pi := \pi_1(X)$. We write $(R^k)'$ when we think of R^k as equipped with the right $\mathbb{Z}[\pi]$-module structure given by $v \cdot g := \beta(g^{-1})v$ for $v \in R^k$ and $g \in \pi$. By Poincaré duality we have (recall that $\widetilde{\partial X}$ is the pre-image of ∂X under the covering map $\tilde{X} \to X$)

$$H_{n-i}^{\beta}(X; R^k) \cong H_{\beta}^i(X, \partial X; (R^k)') := H_i(\text{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{X}, \widetilde{\partial X}), (R^k)')).$$

Using the inner product we get an isomorphism of R-module chain complexes:

$$\text{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{X}, \widetilde{\partial X}), (R^k)') \to \text{Hom}_R(C_*^{\overline{\beta}}(\tilde{X}, \widetilde{\partial X}; R^k), R) = \text{Hom}_R(C_*(\tilde{X}, \widetilde{\partial X}) \otimes_{\mathbb{Z}[\pi]} R^k, R)$$
$$f \mapsto ((c \otimes w) \mapsto \langle f(c), w \rangle).$$

The lemma now follows from applying the universal coefficient theorem. $\square$

2.4. Twisted Alexander polynomials of 3-manifolds

Lemma 2.4. *Let $\phi \in H^1(M)$ be non-trivial and $\alpha : \pi_1(M) \to \text{GL}(\mathbb{F}, k)$ a representation. Then*

(1) $\Delta_0^{\alpha}(t) \neq 0$,
(2) $\Delta_3^{\alpha}(t) = 1$.

Proof. Both statements follow from a straightforward argument using a cell decomposition of M as in the proof of Proposition 6.1. Alternatively note that Kirk and Livingston showed (1) in [16, Proposition 3.5]. For (2) we apply Lemma 2.3 with $R = \mathbb{F}[t^{\pm 1}]$ and $\beta = \alpha \otimes \phi$, and get

$$H_3^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}]) \cong \text{Hom}_{\mathbb{F}[t^{\pm 1}]}(H_0^{\overline{\alpha \otimes \phi}}(M, \partial M; \mathbb{F}^k[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}])$$

as $\mathbb{F}[t^{\pm 1}]$-modules. Note that $\overline{\alpha \otimes \phi} = \overline{\alpha} \otimes (-\phi)$. It follows from (1) that $H_0^{\overline{\alpha} \otimes (-\phi)}(M; \mathbb{F}^k[t^{\pm 1}])$ is $\mathbb{F}[t^{\pm 1}]$-torsion. It follows from the long exact homology sequence that $H_0^{\overline{\alpha} \otimes (-\phi)}(M, \partial M; \mathbb{F}^k[t^{\pm 1}])$ is $\mathbb{F}[t^{\pm 1}]$-torsion as well, hence $H_3^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}]) = 0$. $\square$

Proposition 2.5. *Let M be a 3-manifold whose boundary is empty or consists of tori and let $\phi \in H^1(M)$ be non-trivial. Let $\alpha : \pi_1(M) \to \text{GL}(\mathbb{F}, k)$ be a representation such that $\Delta_1^{\alpha}(t) \neq 0$.*

(1) *If M is closed, then*
$$\Delta_2^{\alpha}(t) = \Delta_0^{\overline{\alpha}}(t^{-1}).$$

(2) *If M has non-empty boundary, then $\Delta_2^{\alpha}(t) = 1$.*

In particular $\deg(\Delta_2^{\alpha}(t)) = b_3(M) \deg(\Delta_0^{\overline{\alpha}}(t))$. Furthermore, if α is unitary, i.e. $\alpha = \overline{\alpha}$, then $\deg(\Delta_2^{\alpha}(t)) = b_3(M) \deg(\Delta_0^{\alpha}(t))$.

For the proof we need the following two useful lemmas which we will also need several times later.

Lemma 2.6. *Let R be a ring, A a group and $\alpha : A \to \text{GL}(R, k)$ a representation. If $\varphi : B \to A$ is a homomorphism, then $H_0^{\alpha \circ \varphi}(B; R^k) \to H_0^{\alpha}(A; R^k)$ is surjective. Furthermore if φ is an epimorphism, then $H_0^{\alpha \circ \varphi}(B; R^k) \to H_0^{\alpha}(A; R^k)$ is an isomorphism.*

Proof. The lemma follows immediately from the commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
0 & \to & \{\alpha(\varphi(b))v - v \mid b \in B, v \in R^k\} & \to & R^k & \to & H_0^{\alpha \circ \varphi}(B; R^k) & \to & 0 \\
  &   & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \{\alpha(a)v - v \mid a \in A, v \in R^k\} & \to & R^k & \to & H_0^{\alpha}(A; R^k) & \to & 0
\end{array}
$$

and the observation that the vertical map on the left is injective (respectively an isomorphism).   □

A standard argument shows the following lemma.

Lemma 2.7. *Let X be an n-manifold, $\mathbb{K}$ a field (e.g. $\mathbb{F}$ or $\mathbb{F}(t)$), and $\alpha : \pi_1(X) \to \mathrm{GL}(\mathbb{K}, k)$ a representation. Then*

$$
\sum_{i=0}^{n} (-1)^i \dim_{\mathbb{K}}(H_*^{\alpha}(X; \mathbb{K}^k)) = k\chi(X).
$$

Proof of Proposition 2.5. We will first show that $H_2^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}])$ is $\mathbb{F}[t^{\pm 1}]$-torsion. Note that it follows from the long exact homology sequence for $(M, \partial M)$ and from duality that $\chi(M) = \frac{1}{2}\chi(\partial M)$. Hence $\chi(M) = 0$ in our case. It now follows from Lemma 2.7 (applied to the field $\mathbb{F}(t)$) that

$$
\sum_{i=0}^{3} (-1)^i \dim_{\mathbb{F}(t)}(H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}] \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t))) = k \cdot \chi(M) = 0.
$$

Note that $H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}] \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t)) = H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t)$ since $\mathbb{F}(t)$ is flat over $\mathbb{F}[t^{\pm 1}]$. By assumption and by Lemma 2.4 we have $H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t) = 0$ for $i \neq 2$, hence $H_2^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t) = 0$ as well.

Now we apply Lemma 2.3 and using that $\overline{\alpha \otimes \phi} = \overline{\alpha} \otimes (-\phi)$ we get

$$
H_2^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}]) \cong \mathrm{Hom}_{\mathbb{F}[t^{\pm 1}]}\left(H_1^{\overline{\alpha} \otimes (-\phi)}(M, \partial M; \mathbb{F}^k[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]\right)
$$

$$
\oplus \mathrm{Ext}_{\mathbb{F}[t^{\pm 1}]}\left(H_0^{\overline{\alpha} \otimes (-\phi)}(M, \partial M; \mathbb{F}^k[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]\right)
$$

as $\mathbb{F}[t^{\pm 1}]$-modules. Since we know that $H_2^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}])$ is $\mathbb{F}[t^{\pm 1}]$-torsion it follows that the first summand on the right hand side is zero.

By Lemma 2.4 $H_0^{\overline{\alpha} \otimes (-\phi)}(M; \mathbb{F}^k[t^{\pm 1}])$ is $\mathbb{F}[t^{\pm 1}]$-torsion. From the long exact homology sequence of the pair $(M, \partial M)$ it follows that $H_0^{\overline{\alpha} \otimes (-\phi)}(M, \partial M; \mathbb{F}^k[t^{\pm 1}])$ is also $\mathbb{F}[t^{\pm 1}]$-torsion. Since $H_0^{\overline{\alpha} \otimes (-\phi)}(M, \partial M; \mathbb{F}^k[t^{\pm 1}])$ is a finitely generated $\mathbb{F}[t^{\pm 1}]$-torsion module and $\mathbb{F}[t^{\pm 1}]$ is a PID, $\mathrm{Ext}_{\mathbb{F}[t^{\pm 1}]}(H_0^{\overline{\alpha \otimes \phi}}(M, \partial M; \mathbb{F}^k[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]) \cong H_0^{\overline{\alpha} \otimes (-\phi)}(M, \partial M; \mathbb{F}^k[t^{\pm 1}])$.

If M is closed then we get $H_2^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \cong H_0^{\overline{\alpha} \otimes (-\phi)}(M; \mathbb{F}^k[t^{\pm 1}])$. Therefore we deduce that $\Delta_2^{\alpha}(t) = \Delta_0^{\overline{\alpha}}(t^{-1})$. If $\partial M \neq \emptyset$, then by Lemma 2.6 the map $H_0^{\overline{\alpha} \otimes (-\phi)}(\partial M; \mathbb{F}^k[t^{\pm 1}]) \to H_0^{\overline{\alpha} \otimes (-\phi)}(M; \mathbb{F}^k[t^{\pm 1}])$ is surjective, hence $H_0^{\overline{\alpha} \otimes (-\phi)}(M, \partial M; \mathbb{F}^k[t^{\pm 1}]) = 0$. This shows that $H_2^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) = 0$ and hence $\Delta_2^{\alpha}(t) = 1$.   □

Remark. Given a presentation for $\pi_1(M)$ the polynomials $\Delta_1^{\alpha}(t)$ and $\Delta_0^{\alpha}(t)$ can be computed efficiently using Fox calculus (cf. e.g. [4, p. 98], [16]). We point out that because we view $C_*(\tilde{M})$ as a *right* module

over $\mathbb{Z}[\pi_1(M)]$ we need a slightly different definition of Fox derivatives. We refer to [11, Section 6] for details. Proposition 2.5 allows us to compute $\Delta_2^\alpha(t)$ using the algorithm for computing the 0-th twisted Alexander polynomial.

Remark. Let $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ be a unitary representation. Then the inequality in Theorem 1.1 becomes the computationally slightly simpler inequality

$$\|\phi\|_T \geq \frac{1}{k}(\deg(\Delta_1^\alpha(t)) - (1 + b_3(M)) \deg(\Delta_0^\alpha(t))).$$

2.5. Reidemeister torsion of 3-manifolds

Assume that ∂M is empty or consists of tori and that $\Delta_1^\alpha(t) \neq 0$. Then it follows from Lemma 2.4 and Proposition 2.5 that $\Delta_i^\alpha(t) \neq 0$ for all i and hence $H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}] \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t)) = H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}]) \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t) = 0$ for all i. Therefore the Reidemeister torsion $\tau(M, \phi, \alpha) \in \mathbb{F}(t)^* / \{rt^l \mid r \in \mathbb{F}^*, l \in \mathbb{Z}\}$ is defined. We refer to [26] for an excellent introduction into the theory of Reidemeister torsion.

The following lemma follows from [26, p. 20] combined with the fact that $\Delta_3^\alpha(t) = 1$ (cf. also [16, Theorem 3.4]).

Lemma 2.8. *If* $\Delta_1^\alpha(t) \neq 0$*, then* $\tau(M, \phi, \alpha)$ *is defined and*

$$\tau(M, \phi, \alpha) = \prod_{i=0}^{2} \Delta_i^\alpha(t)^{(-1)^{i+1}} \in \mathbb{F}(t)$$

up to multiplication by a unit in $\mathbb{F}[t^{\pm 1}]$.

For our purposes we can also use this equality as a definition for $\tau(M, \phi, \alpha)$. We will mostly use $\tau(M, \phi, \alpha)$ as a convenient and concise way to store information. We point out that $\tau(M, \phi, \alpha)$ can also be computed directly from the chain complex of M (cf. [26]).

3. Proof of Theorems 1.1 and 1.2

3.1. Proof of Theorem 1.1

For the remainder of this section let M be a 3-manifold and let $\phi \in H^1(M)$ be primitive. A *weighted surface* $\hat{S}$ in M is defined to be a collection of pairs (S_i, w_i), $i = 1, \ldots, l$ where $S_i \subset M$ are properly disjointly embedded, oriented surfaces in M and w_i are positive integers. We denote the union $\bigcup_i S_i \subset M$ by S'.

Every weighted surface $\hat{S}$ defines an element $\phi_{\hat{S}} := \sum_{i=1}^{l} w_i \cdot PD([S_i]) \in H^1(M)$ where $PD(f) \in H^1(M)$ denotes the Poincaré dual of an element $f \in H_2(M, \partial M)$. By taking w_i parallel copies of S_i we get an (unweighted) properly embedded oriented surface S such that $\phi_{\hat{S}} = PD([S])$. We need the following very useful proposition proved by Turaev in [28].

Proposition 3.1. *There exists a weighted surface* $\hat{S} = (S_i, w_i)_{i=1,\ldots,l}$ *with*

(1) $\phi_{\hat{S}} = \phi$,

(2) $\chi_-(S) = \|\phi\|_T$, and
(3) $M \setminus S'$ connected.

For the remainder of this section let $\hat{S} = (S_i, w_i)_{i=1,\ldots,l}$ be a weighted surface as in Proposition 3.1. We now do the following:

(1) We pick orientation preserving disjoint embeddings $\iota : S_i \times [0, w_i] \to M, i = 1, \ldots, l$ such that ι restricted to $S_i \times 0$ is the identity (where we identify $S_i \times 0$ with S_i). We identify the image of ι with $S_i \times [0, w_i]$.
(2) Note that with these conventions we have $S = \cup_{i=1}^{l} \cup_{j=0}^{w_i-1} S_i \times j$ and $S' = \cup_{i=1}^{l} S_i$.
(3) For any subset $I \subset [0, 1]$ we write $S \times I = \cup_{i=1}^{l} \cup_{j=0}^{w_i-1} S_i \times (j + I)$.
(4) We let $\epsilon = \frac{1}{2}$.
(5) We let $N = M \setminus S \times (0, \epsilon)$ and we let $N' = M \setminus \cup_{i=1}^{l} S_i \times (0, w_i - 1 + \epsilon)$. Note that N' is connected by Proposition 3.1(3).
(6) We pick a base point m_0 for N' which also serves as a base point for M.
(7) For $i = 1, \ldots, l$ pick a base point s_i of S_i and we pick a path in N' connecting m_0 to s_i.

Recall that given a representation $\beta : \pi_1(M, m_0) \to \mathrm{GL}(R, k)$ we get, using the paths chosen above, induced representations (and hence twisted homology groups) for N' and $S_i \times 0, i = 1, \ldots, l$ which we denote by the same symbol. We will sometimes use the isomorphisms of Lemma 2.1 induced by the chosen paths to identify the twisted homology groups with the twisted subspace homology groups.

In the following let $p : \tilde{M} \to M$ be the universal cover of M corresponding to the base point m_0 as in Section 2.1, in particular $\tilde{M}$ is the set of homotopy classes of paths in M starting at m_0. Also we again write $\tilde{X} = p^{-1}(X) \subset \tilde{M}$ for any $X \subset M$. Now note that given $a, a + \delta \in [0, w_i]$ we get a $\mathbb{Z}[\pi_1(M, m_0)]$-equivariant map $f_\delta : p^{-1}(S_i \times a) \to p^{-1}(S_i \times (a + \delta))$ by extending a path from m_0 to a point in $S_i \times a$ in the obvious way to a path from m_0 to a point in $S_i \times (a + \delta)$. In particular we get an induced map

$$f_\delta : H_i^\beta(S_j \times a \subset M; R^k) \to H_i^\beta(S_j \times (a + \delta) \subset M; R^k).$$

Using our choices of paths and using Lemma 2.1 we get a map

$$H_i^\beta(S_j; R^k) = H_i^\beta(S_j \times 0; R^k) \overset{\cong}{\to} H_i^\beta(S_j \times 0 \subset M; R^k) \to H_i^\beta(N' \subset M; R^k) \overset{\cong}{\to} H_i^\beta(N'; R^k)$$

which we denote by ι_-. Similarly we get a map

$$
\begin{aligned}
H_i^\beta(S_j; R^k) = {} & H_i^\beta(S_j \times 0; R^k) \\
& \overset{\cong}{\to} H_i^\beta(S_j \times 0 \subset M; R^k) \xrightarrow{f_{w_j-1+\epsilon}} H_i^\beta(S_j \times (w_j - 1 + \epsilon) \subset M; R^k) \\
& \to H_i^\beta(N' \subset M; R^k) \overset{\cong}{\to} H_i^\beta(N'; R^k)
\end{aligned}
$$

which we denote by ι_+.

For the remainder of this section we pick a representation $\alpha : \pi_1(M, m_0) \to \mathrm{GL}(\mathbb{F}, k)$. With our conventions and choices we can now formulate the following crucial lemma which provides the link between the twisted homology of $S_1, \ldots, S_l$ and the homology of M.

Proposition 3.2. *There exists a long exact sequence*

$$\cdots \to \bigoplus_{j=1}^{l} H_i^{\alpha}(S_j; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \xrightarrow{\overset{l}{\underset{j=1}{\bigoplus}} \iota_- - \iota_+ t^{w_j}} H_i^{\alpha}(N'; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \to H_i^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}]) \to \cdots$$

Proof. We have the following Mayer–Vietoris sequence of twisted subspace homology (where we write $V = \mathbb{F}^k[t^{\pm 1}]$):

$$\to \begin{matrix} H_i^{\alpha \otimes \phi}(S \times \epsilon \subset M; V) \\ \oplus \\ H_i^{\alpha \otimes \phi}(S \times 0 \subset M; V) \end{matrix} \xrightarrow{\left(\begin{smallmatrix} \iota & \iota \\ -\iota & -\iota \end{smallmatrix}\right)} \begin{matrix} H_i^{\alpha \otimes \phi}(N \subset M; V) \\ \oplus \\ H_i^{\alpha \otimes \phi}(S \times [0, \epsilon] \subset M; V) \end{matrix} \xrightarrow{(\iota\;\;\iota)} H_i^{\alpha \otimes \phi}(M; V) \to$$

where ι stands for the maps induced by the various injections. Now consider the following commutative diagram of sequences

$$\begin{matrix} H_i^{\alpha \otimes \phi}(S \times \epsilon \subset M; V) \\ \oplus \\ H_i^{\alpha \otimes \phi}(S \times 0 \subset M; V) \\ \uparrow \left(\begin{smallmatrix} -f_\epsilon \\ \mathrm{id} \end{smallmatrix}\right) \\ H_i^{\alpha \otimes \phi}(S \times 0 \subset M; V) \end{matrix} \quad \begin{matrix} \xrightarrow{\left(\begin{smallmatrix} \iota & \iota \\ -\iota & -\iota \end{smallmatrix}\right)} \\ \\ \\ \xrightarrow{\iota - \iota \circ f_\epsilon} \end{matrix} \quad \begin{matrix} H_i^{\alpha \otimes \phi}(N \subset M; V) \\ \oplus \\ H_i^{\alpha \otimes \phi}(S \times [0, \epsilon] \subset M; V) \\ \uparrow \left(\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right) \\ H_i^{\alpha \otimes \phi}(N \subset M; V) \end{matrix} \quad \begin{matrix} \xrightarrow{(\iota\;\;\iota)} H_i^{\alpha \otimes \phi}(M; V) \\ \\ \\ \\ \uparrow \\ \xrightarrow{\iota} H_i^{\alpha \otimes \phi}(M; V). \end{matrix}$$

Note that given $a \in H_i^{\alpha \otimes \phi}(S \times \epsilon \subset M; \mathbb{F}^k[t^{\pm 1}])$ and $b \in H_i^{\alpha \otimes \phi}(S \times 0 \subset M; \mathbb{F}^k[t^{\pm 1}])$ we have $\iota(a) + \iota(b) = 0 \in H_i^{\alpha \otimes \phi}(S \times [0, \epsilon] \subset M; \mathbb{F}^k[t^{\pm 1}])$ if and only if $a = -f_\epsilon(b)$. From this it now follows easily that the bottom sequence is also exact.

Now note that we have canonical isomorphisms

$$H_i^{\alpha \otimes \phi}(N \subset M; V) \cong H_i^{\alpha \otimes \phi}(N' \subset M; V) \oplus \bigoplus_{j=1}^{l} \bigoplus_{s=1}^{w_j - 1} H_i^{\alpha \otimes \phi}(S_j \times [s-1+\epsilon, s] \subset M; V)$$

$$H_i^{\alpha \otimes \phi}(S \subset M; V) \cong \bigoplus_{j=1}^{l} H_i^{\alpha \otimes \phi}(S_j \times 0 \subset M; V) \oplus \bigoplus_{j=1}^{l} \bigoplus_{s=1}^{w_j - 1} H_i^{\alpha \otimes \phi}(S_j \times s \subset M; V).$$

It is now easy to see, using the same arguments as above, that the following sequence is exact as well:

$$\to \bigoplus_{j=1}^{l} H_i^{\alpha \otimes \phi}(S_j \times 0 \subset M; V) \xrightarrow{\overset{l}{\underset{j=1}{\bigoplus}} \iota - \iota \circ f_{w_j - 1 + \epsilon}} H_i^{\alpha \otimes \phi}(N' \subset M; V) \xrightarrow{\iota} H_i^{\alpha \otimes \phi}(M; V) \to \cdots.$$

Now note that ϕ vanishes on $H_1(N')$ and on every $H_1(S_j)$. Indeed, every curve in S_j can be pushed off into N', where ϕ vanishes. We therefore get *canonical* isomorphisms $H_i^{\alpha \otimes \phi}(N'; \mathbb{F}^k[t^{\pm 1}]) \cong H_i^{\alpha}(N'; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}]$ and $H_i^{\alpha \otimes \phi}(S_j; \mathbb{F}^k[t^{\pm 1}]) \cong H_i^{\alpha}(S_j; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}]$. We are done once we prove the following claim.

Claim. *The diagram*

$$
\begin{array}{ccc}
H_i^{\alpha\otimes\phi}(S_j \times 0 \subset M; \mathbb{F}^k[t^{\pm1}]) & \xrightarrow{\iota - \iota \circ f_{w_j-1+\epsilon}} & H_i^{\alpha\otimes\phi}(N' \subset M; \mathbb{F}^k[t^{\pm1}]) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
H_i^{\alpha\otimes\phi}(S_j \times 0; \mathbb{F}^k[t^{\pm1}]) & \longrightarrow & H_i^{\alpha\otimes\phi}(N'; \mathbb{F}^k[t^{\pm1}]) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
H_i^{\alpha}(S_j; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm1}] & \xrightarrow{\iota_- - \iota_+ t^{w_j}} & H_i^{\alpha}(N'; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm1}]
\end{array}
$$

commutes. Here the top vertical maps are given by the isomorphisms induced from the choice of paths, and the bottom isomorphisms are the canonical isomorphisms mentioned above.

In order to prove the claim first recall that $\tilde{M}$ can be viewed as the homotopy classes of paths in M starting at m_0. Since ϕ vanishes on N' we can define $\phi : \tilde{N}' \to \mathbb{Z}$ (recall that $\tilde{N}' \subset \tilde{M}$) by sending $q \in \tilde{N}'$ (represented by a path which we also call q) to ϕ of the closed path given by juxtaposing q with a path in N' from the end point of q to m_0. This is a well-defined surjective map and we can decompose $\tilde{N}' = \cup_{r\in\mathbb{Z}} \tilde{N}'_r$ where $\tilde{N}'_r = \phi^{-1}(r)$. Now note that the isomorphism from Lemma 2.1 (applied to the constant path which connects the base point m_0 of N' with the base point m_0 of M) gives an isomorphism

$$
H_i((\oplus_r C_*(\tilde{N}'_r)) \otimes_{\mathbb{Z}[\pi_1(M,m_0)]} \mathbb{F}^k[t^{\pm1}]) = H_i^{\alpha\otimes\phi}(N' \subset M; \mathbb{F}^k[t^{\pm1}])
$$
$$
\cong H_i^{\alpha\otimes\phi}(N'; \mathbb{F}^k[t^{\pm1}])
$$
$$
\cong H_i^{\alpha}(N'; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm1}]
$$

where an element represented by $\sigma_r \otimes vt^l$ with $\sigma_r \in C_*(\tilde{N}'_r)$ and $v \in \mathbb{F}^k$ gets sent to an element of the form $\sigma \otimes t^{r+l}$ where $\sigma \in H_i^{\alpha}(N'; \mathbb{F}^k)$.

Similarly we can decompose $\widetilde{S_j \times 0} = \cup_{r\in\mathbb{Z}} \widetilde{(S_j \times 0)}_r$ (using paths in N' again), and the same arguments apply.

Now note that the inclusion $\widetilde{S_j \times 0} \to \tilde{N}'$ clearly sends $\widetilde{(S_j \times 0)}_r$ into $\tilde{N}'_r$. On the other hand the map $f_{w_j-1+\epsilon} : \widetilde{S_j \times 0} \to \tilde{N}'$ sends a point in $\widetilde{S_j \times 0}$ represented by a path γ to the the point represented by the extension of the path γ through $S_j \times [0, w_j - 1 + \epsilon]$. Closing it up in N' we get a path whose intersection number with S is increased by w_j. This shows that $f_{w_j-1+\epsilon}$ sends $\widetilde{(S_j \times 0)}_r$ into $\tilde{N}'_{r+w_j}$. The claim now follows easily from the above observations. $\square$

In the following we write $b_n^{\alpha}(S) := \dim_{\mathbb{F}}(H_n^{\alpha}(S; \mathbb{F}^k)) = \sum_{i=1}^l w_i \dim_{\mathbb{F}}(H_n^{\alpha}(S_i; \mathbb{F}^k))$.

Proposition 3.3. *We have*

$$
b_1^{\alpha}(S) \geq \dim_{\mathbb{F}}(\mathrm{Tor}_{\mathbb{F}[t^{\pm1}]}(H_1^{\alpha\otimes\phi}(M; \mathbb{F}^k[t^{\pm1}]))).
$$

In particular if $\Delta_1^{\alpha}(t) \neq 0$, then $b_1^{\alpha}(S) \geq \deg(\Delta_1^{\alpha}(t))$.

The proof is a variation on a standard argument.

Proof. Consider the exact sequence from Proposition 3.2. Note that

$$F := \mathrm{Ker} \left\{ \bigoplus_{j=1}^{l} H_0^\alpha(S_j; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \right.$$

$$\left. \rightarrow H_0^\alpha(N'; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \right\} \subset \bigoplus_{j=1}^{l} H_0^\alpha(S_j; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}]$$

is a (possibly trivial) free $\mathbb{F}[t^{\pm 1}]$-module. Consider the exact sequence

$$\bigoplus_{j=1}^{l} H_1^\alpha(S_j; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \xrightarrow{\bigoplus_{j=1}^{l} \iota_- - \iota_+ t^{w_j}} H_1^\alpha(N'; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \rightarrow H_1^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}]) \xrightarrow{\partial} F \rightarrow 0.$$

Since $\mathbb{F}[t^{\pm 1}]$ is a PID the map ∂ splits, i.e., $H_1^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}]) \cong \mathrm{Ker}(\partial) \oplus F$. In particular

$$\dim_{\mathbb{F}}(\mathrm{Tor}_{\mathbb{F}[t^{\pm 1}]}(H_1^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}]))) = \dim_{\mathbb{F}}(\mathrm{Tor}_{\mathbb{F}[t^{\pm 1}]}(\mathrm{Ker}(\partial))).$$

Using appropriate bases the map $\bigoplus_{j=1}^{l} \iota_- - \iota_+ t^{w_j}$, which represents the module $\mathrm{Ker}(\partial)$, is presented by a matrix of the form

$$\begin{pmatrix} A_1 t^{w_1} + B_1 & \dots & A_l t^{w_l} + B_l \end{pmatrix}$$

where $A_j, B_j, j = 1, \dots, l$ are matrices over $\mathbb{F}$ of size $\dim_{\mathbb{F}}(H_1^\alpha(N'; \mathbb{F}^k)) \times \dim_{\mathbb{F}}\left(H_1^\alpha(S_j; \mathbb{F}^k)\right)$. The proposition now follows easily from combining Lemma 2.2 with the following claim.

Claim. *Let H be a $\mathbb{F}[t^{\pm 1}]$-module with a presentation matrix of the form*

$$C = \begin{pmatrix} A_1 t^{w_1} + B_1 & \dots & A_l t^{w_l} + B_l \end{pmatrix}$$

where A_j, B_j are matrices over $\mathbb{F}$ of size $p \times q_j$. Then $\dim_{\mathbb{F}}(\mathrm{Tor}_{\mathbb{F}[t^{\pm 1}]}(H)) \leq \sum_{j=1}^{l} q_j w_j$.

For the proof of the claim let $q = \sum_{j=1}^{l} q_j$. Using row and column operations over the PID $\mathbb{F}[t^{\pm 1}]$ we can transform C into a matrix of the form

$$\begin{pmatrix} f_1(t) & 0 & \dots & 0 & 0 \\ 0 & f_2(t) & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & f_r(t) & 0 \\ 0 & 0 & \dots & 0 & (0)_{p-r \times q-r} \end{pmatrix}$$

for some $f_i(t) \in \mathbb{F}[t^{\pm 1}] \setminus \{0\}$. Clearly $\dim_{\mathbb{F}}(\mathrm{Tor}_{\mathbb{F}[t^{\pm 1}]}(H)) = \sum_{i=1}^{r} \deg(f_i(t))$. Since row and column operations do not change the ideals of $\mathbb{F}[t^{\pm 1}]$ generated by minors (cf. [4, p. 101]), and since any $k \times k$ minor of $At + B$ has degree at most $\sum_{j=1}^{l} q_j w_j$, it follows that $\sum_{i=1}^{r} \deg(f_i(t)) \leq \sum_{j=1}^{l} q_j w_j$. This concludes the proof of the claim. $\square$

Proposition 3.4. *If $\Delta_1^\alpha(t) \neq 0$ then either S is connected or $b_0^\alpha(S_j) = 0$ for $j = 1, \dots, l$.*

The following proof is partly inspired by ideas of Turaev [28].

Proof. Consider the exact sequence from Proposition 3.2:

$$\to \quad H_1^{\alpha\otimes\phi}(M;\mathbb{F}^k[t^{\pm 1}])$$

$$\to \bigoplus_{j=1}^{l} H_0^{\alpha}(S_j;\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm 1}] \xrightarrow{\bigoplus_{j=1}^{l}\iota_- - \iota_+ t^{w_j}} H_0^{\alpha}(N';\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm 1}] \to \quad H_0^{\alpha\otimes\phi}(M;\mathbb{F}^k[t^{\pm 1}]) \to 0.$$

From $\Delta_1^{\alpha}(t) \neq 0$ it follows that $H_1^{\alpha\otimes\phi}(M;\mathbb{F}^k[t^{\pm 1}])$ is $\mathbb{F}[t^{\pm 1}]$-torsion. By Lemma 2.4 $H_0^{\alpha\otimes\phi}(M;\mathbb{F}^k[t^{\pm 1}])$ is $\mathbb{F}[t^{\pm 1}]$-torsion. The exact sequence shows that the ranks of the free $\mathbb{F}[t^{\pm 1}]$-modules $\oplus_{j=1}^{l} H_0^{\alpha}(S_j;\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm 1}]$ and $H_0^{\alpha}(N';\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm 1}]$ are equal, and hence

$$\bigoplus_{j=1}^{l} H_0^{\alpha}(S_j;\mathbb{F}^k) \cong H_0^{\alpha}(N';\mathbb{F}^k). \tag{3}$$

Note that the representations $\pi_1(S_j, s_j) \to \pi_1(M, m_0) \xrightarrow{\alpha} \mathrm{GL}(\mathbb{F}, k)$ induced by our chosen paths factor through $\pi_1(N', m_0)$. Therefore

$$b_0^{\alpha}(S_j) \geq b_0^{\alpha}(N'), \quad j = 1, \ldots, l \tag{4}$$

by Lemma 2.6.

First consider the case $b_0^{\alpha}(N') = 0$. In that case it follows from isomorphism (3) that $b_0^{\alpha}(S_j) = 0$ for all $j = 1, \ldots, l$.

Now assume that $b_0^{\alpha}(N') > 0$. It follows immediately from the isomorphism in (3) and from the inequality (4) that $l = 1$. But since ϕ is primitive it also follows that $w_1 = 1$, i.e., S is connected.   □

Proposition 3.5. *Assume that $\Delta_1^{\alpha}(t) \neq 0$, then*

$$b_0^{\alpha}(S) = \deg(\Delta_0^{\alpha}(t)).$$

Proof. First assume that $b_0^{\alpha}(S_j) = 0$ for every component $j = 1, \ldots, l$. Then $H_0^{\alpha}(N';\mathbb{F}^k) = 0$ by the isomorphism (3). This implies that $H_0^{\alpha\otimes\phi}(M;\mathbb{F}^k[t^{\pm 1}]) = 0$ from the exact sequence in Proposition 3.2, hence $\Delta_0^{\alpha}(t) = 1$.

Now assume that $b_0^{\alpha}(S_j) \neq 0$ for some j. By Proposition 3.4 S is connected and hence $N' = N$. It follows from Lemma 2.6 that the maps $\iota_+, \iota_- : H_0^{\alpha}(S;\mathbb{F}^k) \to H_0^{\alpha}(N';\mathbb{F}^k)$ are surjective. Since $H_0^{\alpha}(S;\mathbb{F}^k) \cong H_0^{\alpha}(N';\mathbb{F}^k)$ by isomorphism (3) it follows that ι_+ and ι_- induce isomorphisms on $H_0^{\alpha}(S;\mathbb{F}^k)$.

Let $b := b_0^{\alpha}(S) = b_0^{\alpha}(N')$. Picking appropriate bases for $H_0^{\alpha}(S;\mathbb{F}^k)$ and $H_0^{\alpha}(N';\mathbb{F}^k)$ the sequence from Proposition 3.2 becomes

$$\mathbb{F}^b \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \xrightarrow{t \cdot Id - J} \mathbb{F}^b \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \to H_0^{\alpha\otimes\phi}(M;\mathbb{F}^k[t^{\pm 1}]) \to 0,$$

where $J : \mathbb{F}^b \to \mathbb{F}^b$ is an isomorphism. It follows that $H_0^{\alpha\otimes\phi}(M;\mathbb{F}^k[t^{\pm 1}]) \cong \mathbb{F}^b \cong H_0^{\alpha}(S;\mathbb{F}^k)$. The lemma now follows from Lemma 2.2.   □

Proposition 3.6. *Assume that ∂M is empty or consists of tori. If $\Delta_1^{\alpha}(t) \neq 0$, then*

$$b_2^{\alpha}(S) = \deg(\Delta_2^{\alpha}(t)).$$

Proof. Let $I := \{i \in \{1, \ldots, l\} \mid S_i \text{ closed}\}$ and let $T = \cup_{i \in I} S_i$. Clearly $b_2^\alpha(S) = \sum_{i \in I} w_i b_2^\alpha(S_i)$. Note that we can write $\partial N' = T_+ \cup T_- \cup W$ for some surface W where $T_- = T \times 0$ and $T_+ = \cup_{i \in I} S_i \times (w_i - 1 + \epsilon)$. It follows from Lemma 2.4 and Proposition 2.5 that the long exact sequence from Proposition 3.2 becomes

$$0 \to \bigoplus_{i \in I} H_2^\alpha(S_i; \mathbb{F}^k) \otimes_\mathbb{F} \mathbb{F}[t^{\pm 1}] \xrightarrow{\overset{\oplus \iota_- - \iota_+ t^{w_i}}{i \in I}} H_2^\alpha(N'; \mathbb{F}^k) \otimes_\mathbb{F} \mathbb{F}[t^{\pm 1}] \to H_2^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}]) \to 0.$$

We need the following claim.

Claim. *The maps $\iota_-, \iota_+ : H_2^\alpha(T; \mathbb{F}^k) = \oplus_{i \in I} H_2^\alpha(S_i; \mathbb{F}^k) \to H_2^\alpha(N'; \mathbb{F}^k)$ are isomorphisms.*

In order to give a proof of the claim we first tensor the above short exact sequence with $\mathbb{F}(t)$. We see that $H_2^\alpha(T; \mathbb{F}^k)$ and $H_2^\alpha(N'; \mathbb{F}^k)$ have the same dimension as $\mathbb{F}$-vector spaces. It is therefore enough to show that ι_- and ι_+ are injections. This is clearly the case if $T = \emptyset$. So let us now assume that $T \neq \emptyset$.

Consider the short exact sequence

$$H_3^\alpha(N', T_+; \mathbb{F}^k) \to H_2^\alpha(T_+; \mathbb{F}^k) \to H_2^\alpha(N'; \mathbb{F}^k).$$

Note that $\partial N'$ is the disjoint union of T_+ and $T_- \cup W$ since T_+ is closed. We can therefore apply Poincaré duality. By Poincaré duality and by Lemma 2.3 in Section 2.3 we then have

$$H_3^\alpha(N', T_+; \mathbb{F}^k) \cong H_\alpha^0(N', T_- \cup W; \mathbb{F}^k) \cong \mathrm{Hom}_\mathbb{F}(H_0^{\overline{\alpha}}(N', T_- \cup W; \mathbb{F}^k), \mathbb{F}).$$

Here $\overline{\alpha}$ is the adjoint representation of α which is defined in Section 2.3. Since $T \neq \emptyset$ it follows from Lemma 2.6 that the map $H_0^{\overline{\alpha}}(T_- \cup W; \mathbb{F}^k) \to H_0^{\overline{\alpha}}(N'; \mathbb{F}^k)$ is surjective. It now follows from the long exact homology sequence that $H_0^{\overline{\alpha}}(N', T_- \cup W; \mathbb{F}^k) = 0$. This shows that ι_+ is injective. The proof for ι_- is identical. This concludes the proof of the claim.

We now show that $H_2^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}])$ has a presentation matrix of the form $AT + B$ where A, B are invertible matrices over $\mathbb{F}$ and D is a diagonal matrix with $b_2^\alpha(S_i)$ entries t^{w_i} for any $i \in I$. Note that

$$\det(AD + B) = \det(B) + \cdots + \det(A) t^{\overset{\sum w_i b_2^\alpha(S_i)}{i \in I}}.$$

It follows that

$$\dim(H_2^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}])) = \deg(\det(AD + B)) = \sum_{i \in I} w_i b_2^\alpha(S_i) = b_2^\alpha(S). \quad \square$$

Now we can conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. Without loss of generality we can assume that ϕ is primitive since the Thurston norm and the degrees of twisted Alexander polynomials are homogeneous. Let $\hat{S}$ be the weighted surface from Proposition 3.1. By Lemma 2.7 we have

$$\begin{aligned}
\|\phi\|_T &= \max\{0, b_1(S) - (b_0(S) + b_2(S))\} \\
&\geq b_1(S) - (b_0(S) + b_2(S)) \\
&= \frac{1}{k}(b_1^\alpha(S) - (b_0^\alpha(S) + b_2^\alpha(S))).
\end{aligned}$$

The theorem now follows immediately from Proposition 3.3, Lemma 2.4, Propositions 3.5 and 3.6 and Lemma 2.8.

3.2. Proof of Theorem 1.2

Proof of Theorem 1.2. Let S be a fiber of the fiber bundle $M \to S^1$. Clearly S is dual to $\phi \in H^1(M)$ and it is well-known that S is Thurston norm minimizing. Denote by $\hat{M}$ the infinite cyclic cover of M corresponding to ϕ. Then an easy argument shows that $H_i^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}]) \cong H_i^{\alpha}(\hat{M}; \mathbb{F}^k)$ (cf. also [16, Theorem 2.1]). In particular $H_i^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}]) \cong H_i^{\alpha}(S; \mathbb{F}^k)$.

By assumption $S \neq D^2$ and $S \neq S^2$. Therefore by Lemmas 2.7 and 2.2 we get

$$
\begin{aligned}
\|\phi\|_T &= \chi_-(S) \\
&= b_1(S) - b_0(S) - b_2(S) \\
&= \frac{1}{k}(b_1^{\alpha}(S) - b_0^{\alpha}(S) - b_2^{\alpha}(S)) \\
&= \frac{1}{k}(\dim_{\mathbb{F}}(H_1^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}])) - \dim_{\mathbb{F}}(H_0^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}])) \\
&\quad - \dim_{\mathbb{F}}(H_2^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}]))) \\
&= \frac{1}{k}(\deg(\Delta_1^{\alpha}(t)) - \deg(\Delta_0^{\alpha}(t)) - \deg(\Delta_2^{\alpha}(t))) \\
&= \deg(\tau(M, \phi, \alpha)). \quad \square
\end{aligned}
$$

Since $\|\phi\|_T$ might be unknown for a given example the following corollary to Theorem 1.2 gives sometimes a more practical fibering obstruction.

Corollary 3.7. *Let M be a 3-manifold and $\phi \in H^1(M)$ primitive such that (M, ϕ) fibers over S^1 and such that $M \neq S^1 \times D^2$, $M \neq S^1 \times S^2$. Let $\mathbb{F}$ and $\mathbb{F}'$ be fields. Consider the untwisted Alexander polynomial $\Delta_1(t) \in \mathbb{F}[t^{\pm 1}]$. For any representation $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}', k)$ we have*

$$
\deg(\Delta_1(t)) - (1 + b_3(M)) = \frac{1}{k}(\deg(\Delta_1^{\alpha}(t)) - \deg(\Delta_0^{\alpha}(t)) - \deg(\Delta_2^{\alpha}(t))).
$$

Proof. The corollary follows immediately from applying Theorem 1.2 to the trivial representation $\pi_1(M) \to \mathrm{GL}(\mathbb{F}, 1)$ and to the representation α. $\quad \square \quad \square$

4. The case of vanishing Alexander polynomials

Let L be a boundary link (for example a split link). It is well-known that the untwisted one-variable and multi-variable Alexander polynomials of L vanish (cf. [14]). Similarly one can see that in fact most of the twisted one-variable and multi-variable Alexander polynomials vanish as well. (See [7] for the definition of twisted multi-variable Alexander polynomials.) Therefore Theorem 1.1 can in most cases not be applied to get lower bounds on the Thurston norm.

It follows clearly from Propositions 3.3 and 3.6 that the condition $\Delta_1^{\alpha}(t) \neq 0$ is only needed to ensure that there exists a surface S dual to ϕ with $b_0^{\alpha}(S) = \deg(\Delta_0^{\alpha}(t))$ and $b_2^{\alpha}(S) = \deg(\Delta_2^{\alpha}(t))$. The following theorem can often be applied in the case of link complements.

Theorem 4.1. *Let M be a 3-manifold such that $H^1(M) \xrightarrow{i^*} H^1(\partial M)$ is an injection where i^* is the inclusion-induced homomorphism. Let N be a torus component of ∂M, $\phi \in H^1(N) \cap \mathrm{Im}(i^*)$ primitive, and $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ a representation. Then*

$$\|(i^*)^{-1}(\phi)\|_T \geq \frac{1}{k} \deg(\tilde{\Delta}_1^\alpha(t)) - 1.$$

It is not hard to show that we can find a Thurston norm minimizing surface dual to $(i^*)^{-1}(\phi)$ which is connected and has boundary (cf. e.g. [11, Corollary 10.4] or Turaev [28, p. 14]). The theorem now follows from the proof of Theorem 1.1.

The main application is to study the Thurston norm of the complement of a link $L = L_1 \cup \cdots \cup L_m \subset S^3$. In this case we can take ϕ to be dual to the meridian of the ith component L_i. Then it follows from the proof of Theorem 4.1 and a standard argument that $\|(i^*)^{-1}(\phi)\|_T = 2 \operatorname{genus}(L_i) - 1$, where $\operatorname{genus}(L_i)$ denotes the minimal genus of a surface in $X(L)$ bounding a parallel copy of L_i. Similar results were obtained by Turaev [28, p. 14] and Harvey [11, Corollary 10.4].

The following observation will show that in more complicated cases there is no immediate way to determine $b_0(S)$: if $L = L_1 \cup L_2$ is a split oriented link, and $\phi : H_1(X(L)) \to \mathbb{Z}$ given by sending the meridians to 1, then a Thurston norm minimizing surface S dual to ϕ is easily seen to be the disjoint union of the Seifert surfaces of L_1 and L_2. On the other hand if L_1 and L_2 are parallel copies of a knot with opposite orientations and $\phi : H_1(X(L)) \to \mathbb{Z}$ is again given by sending the meridians to 1, then the annulus S between L_1 and L_2 is dual to ϕ with Euler characteristic zero. Summarizing, we have two situations in which the first twisted Alexander polynomials vanish (in fact $H_1(X(L); \mathbb{Q}[t^{\pm 1}])$) has rank one), ϕ is of the same type, but $b_0(S)$ differs.

5. Examples

5.1. Representations of 3-manifold groups

In our applications we first find homomorphisms $\pi_1(M) \to S_k$, and then find a representation of S_k. Here S_k denotes the permutation group of order k. The first choice of a representation for S_k that comes to mind is $S_k \to \mathrm{GL}(\mathbb{F}, k)$ where S_k acts by permuting the coordinates. But S_k leaves the subspace $\{(v, v, \ldots, v) \mid v \in \mathbb{F}\} \subset \mathbb{F}^k$ invariant, hence this representation is 'not completely non-trivial'. To avoid this we prefer to work with a slightly different representation of S_k. If $\varphi : \pi_1(M) \to S_k$ is a homomorphism then we consider

$$\alpha(\varphi) : \pi_1(M) \xrightarrow{\varphi} S_k \to \mathrm{GL}(V_{k-1}(\mathbb{F})),$$

where

$$V_l(\mathbb{F}) := \left\{ (v_1, \ldots, v_{l+1}) \in \mathbb{F}^{l+1} \,\middle|\, \sum_{i=1}^{l+1} v_i = 0 \right\}.$$

Clearly $\dim_{\mathbb{F}}(V_l(\mathbb{F})) = l$ and S_{l+1} acts on it by permutation.

We point out that the fundamental groups of 3-manifolds for which the geometrization conjecture holds are residually finite (cf. [24,13]). In particular most (or perhaps all) 3-manifolds have many homomorphisms to finite groups, hence to S_k's.
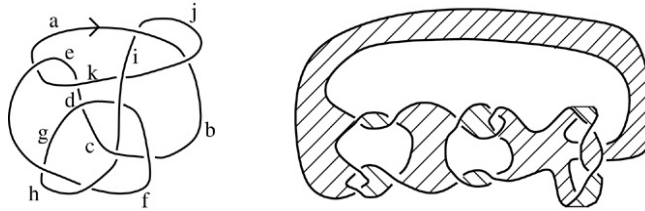
Fig. 1. The Conway knot 11_{401} and a Seifert surface of genus 3 (from [8]).

5.2. Knots with up to 12 crossings: Genus bounds and fiberedness

In this section we show that the degrees of twisted Alexander polynomials detect the genus of all knots with 12 crossings or less. Also we detect all non-fibered knots with 12 crossings or less, some of which are new discoveries to our knowledge.

I. *Knot genus.* There are 36 knots with 12 crossings or less for which genus$(K) > \frac{1}{2} \deg \Delta_K(t)$ (cf. e.g. [2] or [23]). The most famous and interesting examples are $K = 11_{401}$ (the Conway knot) and 11_{409} (the Kinoshita–Terasaka knot). Here we use the *knotscape* notation.

First, we consider the Conway knot $K = 11_{401}$ whose diagram is given in Fig. 1. The genus of the Conway knot is 3. This knot has Alexander polynomial one, i.e., the degree of $\Delta_K(t)$ equals zero. Furthermore this implies that $\pi_1(X(K))^{(1)}$ is perfect, i.e., $\pi_1(X(K))^{(n)} = \pi_1(X(K))^{(1)}$ for any $n > 1$. (For a group G, $G^{(n)}$ is defined inductively as follows: $G^{(0)} := G$ and $G^{(n+1)} := [G^{(n)}, G^{(n)}]$.) Therefore the genus bounds of Cochran [3] and Harvey [11] vanish as well. The fundamental group $\pi_1(X(K))$ is generated by the meridians $a, b, \ldots, k$ of the segments in the knot diagram of Fig. 1. The relations are

$$a = jbj^{-1}, \qquad b = fcf^{-1}, \qquad c = g^{-1}dg, \qquad d = k^{-1}ek,$$
$$e = h^{-1}fh, \qquad f = igi^{-1}, \qquad g = e^{-1}he, \qquad h = c^{-1}ic,$$
$$i = aja^{-1}, \qquad j = iki^{-1}, \qquad k = e^{-1}ae.$$

Using the program *KnotTwister* [5] we found the homomorphism $\varphi : \pi_1(X(K)) \to S_5$ given by

$$a \mapsto (142), \qquad b \mapsto (451), \qquad c \mapsto (451), \qquad d \mapsto (453),$$
$$e \mapsto (453), \qquad f \mapsto (351), \qquad g \mapsto (351), \qquad h \mapsto (431),$$
$$i \mapsto (351), \qquad j \mapsto (352), \qquad k \mapsto (321),$$

where we use cycle notation. We then consider $\alpha := \alpha(\varphi) : \pi_1(X(K)) \xrightarrow{\varphi} S_5 \to \mathrm{GL}(V_4(\mathbb{F}_{13}))$. Using *KnotTwister* we compute $\deg(\Delta_0^\alpha(t)) = 0$ and we compute the first twisted Alexander polynomial to be

$$\Delta_1^\alpha(t) = 1 + 6t + 9t^2 + 12t^3 + t^5 + 3t^6 + t^7 + 3t^8 + t^9 + 12t^{11} + 9t^{12} + 6t^{13} + t^{14} \in \mathbb{F}_{13}[t^{\pm 1}].$$

Theorem 1.1 together with Proposition 2.5 says that if $\Delta_1^\alpha(t) \neq 0$, then

$$\text{genus}(K) \geq \frac{1}{2k} \left(\deg(\Delta_1^\alpha(t)) - \deg(\Delta_0^\alpha(t)) \right) + \frac{1}{2}.$$

Therefore in our case we get

$$\text{genus}(K) \geq \frac{1}{8} \cdot 14 + \frac{1}{2} = \frac{18}{8} = 2.25.$$

Table 1
Computation of degrees of twisted Alexander polynomials

Knotscape name	11_{401}	11_{409}	11_{412}	11_{434}	11_{440}	11_{464}
Genus bound from $\Delta_K(t)$	0	0	2	1	2	1
Genus bound from $\Delta_1^{\alpha}(t)$	2.25	2.00	3.00	2.00	3.00	2.00
Knotscape name	11_{519}	12_{1311}	12_{1316}	12_{1319}	12_{1339}	12_{1344}
Genus bound from $\Delta_K(t)$	2	1	2	1	1	2
Genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	3.00	2.00	2.50	3.00	2.00	3.00
Knotscape name	12_{1351}	12_{1375}	12_{1412}	12_{1417}	12_{1420}	12_{1509}
Genus bound from $\Delta_K(t)$	2	2	1	1	2	2
Genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	3.00	3.00	2.00	3.00	3.00	3.00
Knotscape name	12_{1519}	12_{1544}	12_{1545}	12_{1552}	12_{1555}	12_{1556}
Genus bound from $\Delta_K(t)$	2	2	2	2	2	1
Genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	3.00	3.00	3.00	3.00	3.00	2.00
Knotscape name	12_{1581}	12_{1601}	12_{1609}	12_{1699}	12_{1718}	12_{1745}
Genus bound from $\Delta_K(t)$	1	0	1	1	0	1
Genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	2.00	1.25	2.00	2.00	2.00	2.00
Knotscape name	12_{1807}	12_{1953}	12_{2038}	12_{2096}	12_{2100}	12_{2118}
Genus bound from $\Delta_K(t)$	1	2	2	2	2	2
Genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	2.00	3.00	3.00	3.00	3.00	3.00

Since genus(K) is an integer we get genus(K) ≥ 3. Since there exists a Seifert surface of genus 3 for K (cf. Fig. 1) it follows that the genus of the Conway knot is indeed 3.

Second, let K be the Kinoshita–Terasaka knot $K = 11_{409}$. The genus of K is 2. We found a map $\varphi : \pi_1(X(K)) \to S_5$ such that $\Delta_1^{\alpha(\varphi)}(t) \in \mathbb{F}_{13}[t^{\pm 1}]$ has degree 12 and $\deg(\Delta_0^{\alpha(\varphi)}(t)) = 0$. It follows from Theorem 1.1 that genus(K) $\geq \frac{1}{8} \cdot 12 + \frac{1}{2} = 2$. Note that in this case our inequality becomes equality, hence 'rounding up' is not necessary. Our Table 1 shows that this is surprisingly often the case. This fact is of importance in [7] where we study the Thurston norm of link complements.

Table 1 gives all knots with 12 crossings or less for which $\deg(\Delta_K(t)) < 2\,\text{genus}(K)$. We obtained the list of these knots from Alexander Stoimenow's knot page [23]. We compute twisted Alexander polynomials using *KnotTwister* and 4-dimensional representations of the form $\alpha(\varphi) : \pi_1(X(K)) \xrightarrow{\varphi} S_5 \to \text{GL}(V_4(\mathbb{F}_{13}))$. Our genus bounds from Theorem 1.1 give (by rounding up if necessary) the correct genus in each case.

Using *KnotTwister* it takes only a few seconds to find such representations and to compute the twisted Alexander polynomial.

Remark. Let K_1 and K_2 be knots and assume there exists an epimorphism $\varphi : \pi_1(X(K_1)) \to \pi_1(X(K_2))$. Simon asked (cf. question 1.12(b) on Kirby's problem list [15]) whether this implies that genus(K_1) $\geq$ genus(K_2). Let $\alpha : \pi_1(X(K_2)) \to \text{GL}(\mathbb{F}, k)$ be a representation. By [17] $\Delta_{K_2,1}^{\alpha}(t)$ divides $\Delta_{K_1,1}^{\alpha\circ\varphi}(t)$. Together with Lemma 2.6 this shows that the genus bounds from Theorem 1.1 for K_1 are greater than or equal to the bounds for K_2. Thus Theorem 1.1 (together with the observation that Theorem 1.1 often detects the correct genus) suggests an affirmative answer to Simon's question. This should also be compared to the results in [12].

Table 2
Alexander polynomials of non-fibered knots

Knotscape name	12_{1345}	12_{1498}	12_{1502}	12_{1546}	12_{1567}	12_{1670}	12_{1682}
Order of permutation group k	4	5	5	3	5	5	4
Order p of finite field	3	2	11	2	3	2	3
Knotscape name	12_{1752}	12_{1771}	12_{1823}	12_{1938}	12_{2089}	12_{2103}	
Order of permutation group k	3	3	5	5	5	4	
Order p of finite field	2	7	7	11	2	3	

Remark. There are situations when for a given manifold the degree of the twisted Alexander polynomial for some representation gives a worse bound for the Thurston norm than the degree of the untwisted Alexander polynomial. This should be compared to the situation of [3,12,6]: Cochran's and Harvey's sequence of higher order Alexander polynomials gives a never decreasing sequence of lower bounds on the Thurston norm.

II. *Fiberedness.* Consider the knot $K = 12_{1345}$. Its Alexander polynomial equals $\Delta_K(t) = 1 - 2t + 3t^2 - 2t^3 + t^4$ and its genus equals two, therefore K satisfies Neuwirth's condition (1) in Section 1.4. It follows from Corollary 3.7 that if K were fibered, then for any field $\mathbb{F}$ and any representation $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ the following would hold:

$$\deg(\Delta_K(t)) = \frac{1}{k}(\deg(\Delta_1^\alpha(t)) - \deg(\Delta_0^\alpha(t))) + 1.$$

We found a representation $\alpha : \pi_1(X(K)) \to S_4$ such that for the canonical representation $\alpha : \pi_1(X(K)) \to S_4 \to \mathrm{GL}(\mathbb{F}_3, 4)$ given by permuting the coordinates, we get $\deg(\Delta_1^\alpha(t)) = 7$ and $\deg(\Delta_0^\alpha(t)) = 1$. We compute

$$\frac{1}{4}\deg(\Delta_1^\alpha(t)) - \frac{1}{4}\deg(\Delta_0^\alpha(t)) + 1 = \frac{10}{4} \neq 4 = \deg(\Delta_K(t)).$$

Hence K is not fibered.

Similarly we found altogether thirteen 12-crossings knots which satisfy condition (1) but which are not fibered; we list them in Table 2. As we mentioned in the introduction, Stoimenow and Hirasawa showed that the remaining 12-crossing knots are fibered if and only if they satisfy Neuwirth's condition (1). Altogether this completes the classification of all fibered 12-crossing knots.

5.3. Closed manifolds

Let $K \subset S^3$ be a non-trivial knot, denote the result of zero-framed surgery along K by M_K. Let $\phi \in H^1(M_K)$ be a generator. Gabai [9, Theorem 8.8] showed that for a non-trivial knot K we have $\|\phi\|_{T,M_K} = 2\,\mathrm{genus}(K) - 2$. Furthermore Gabai [9] showed that a knot K is fibered if and only if M_K is fibered.

Using *KnotTwister* one can easily see that, for any knot K with 12 crossings or less, twisted Alexander polynomials corresponding to appropriate representations determine the Thurston norm of M_K and detect whether M_K is fibered or not.

6. Generalization of Cha's fibering obstruction

In this section we formulate and prove Proposition 6.1 which, in combination with Theorem 1.2, immediately implies Theorem 1.3. (In Theorem 1.3 it is easy to prove that if ∂M is non-empty then ∂M has to be a collection of tori.)

In order to formulate Proposition 6.1 we need some more notation. For a ring R and a maximal ideal $\mathfrak{m} \subset R$ we denote the field $R/\mathfrak{m}$ by $\mathbb{F}_\mathfrak{m}$. Furthermore given a representation $\alpha : \pi_1(M) \to \mathrm{GL}(R, k)$ we denote by $\alpha_\mathfrak{m}$ the representation $\pi_1(M) \xrightarrow{\alpha} \mathrm{GL}(R, k) \to \mathrm{GL}(\mathbb{F}_\mathfrak{m}, k)$ where $\mathrm{GL}(R, k) \to \mathrm{GL}(\mathbb{F}_\mathfrak{m}, k)$ is induced from the canonical map $\pi_\mathfrak{m} : R \to R/\mathfrak{m} = \mathbb{F}_\mathfrak{m}$. The main example to keep in mind is $R = \mathbb{Z}$, $\mathfrak{m} = (p)$ for a prime p, and $R/\mathfrak{m} = \mathbb{Z}/(p) = \mathbb{F}_p$.

Proposition 6.1. *Let M be a 3-manifold whose boundary is empty or consists of tori and let R be a Noetherian UFD. Let $\phi \in H^1(M)$ be non-trivial and $\alpha : \pi_1(M) \to \mathrm{GL}(R, k)$ a representation. Then $\Delta_1^\alpha(t) \in R[t^{\pm 1}]$ is monic and*

$$\|\phi\|_T = \frac{1}{k} \deg(\tau(M, \phi, \alpha))$$

if and only if for all maximal ideals $\mathfrak{m}$ of R we have that $\Delta_1^{\alpha_\mathfrak{m}}(t)$ is non-trivial and

$$\|\phi\|_T = \frac{1}{k} \deg(\tau(M, \phi, \alpha_\mathfrak{m})).$$

Proof. We only prove this proposition in the case that M is closed. The proof for the case that ∂M is a non-empty collection of tori is very similar. Note that in either case $\chi(M) = 0$.

We first make use of an argument in the proof of [20, Theorem 5.1]. Choose a triangulation τ of M. Let T be a maximal tree in the 1-skeleton of τ and let T' be a maximal tree in the dual 1-skeleton. We collapse T to form a single 0-cell and join the 3-simplices along T' to form a single 3-cell. Denote the number of 1-cells by n. It follows from M closed that $\chi(M) = 0$, hence there are n 2-cells. From the CW structure we obtain a chain complex $C_* = C_*(\tilde{M})$ of the following form

$$0 \to C_3(\tilde{M}) \xrightarrow{\partial_3} C_2(\tilde{M}) \xrightarrow{\partial_2} C_1(\tilde{M}) \xrightarrow{\partial_1} C_0(\tilde{M}) \to 0$$

where $C_i(\tilde{M}) \cong \mathbb{Z}[\pi_1(M)]$ for $i = 0, 3$ and $C_i(\tilde{M}) \cong \mathbb{Z}[\pi_1(M)]^n$ for $i = 1, 2$. Let $A_i, i = 0, \dots, 3$ over $\mathbb{Z}[\pi_1(M)]$ be the matrices corresponding to the boundary maps $\partial_i : C_i \to C_{i-1}$ with respect to the bases given by the lifts of the cells of M to $\tilde{M}$. We can arrange the lifts such that

$$A_3 = (1 - g_1, 1 - g_2, \dots, 1 - g_n)^t,$$
$$A_1 = (1 - h_1, 1 - h_2, \dots, 1 - h_n).$$

Note that $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_n\}$ are generating sets for $\pi_1(M)$ since M is a closed 3-manifold. Since ϕ is non-trivial there exist r, s such that $\phi(g_r) \neq 0$ and $\phi(h_s) \neq 0$. Let B_3 be the rth row of A_3. Let B_2 be the result of deleting the rth column and the sth row from A_2. Let B_1 be the sth column of A_1.

Given a $p \times q$ matrix $B = (b_{rs})$ be with entries in $\mathbb{Z}[\pi]$ we write $b_{rs} = \sum b_{rs}^g g$ for $b_{rs}^g \in \mathbb{Z}, g \in \pi$. We then define $(\alpha \otimes \phi)(B)$ to be the $p \times q$ matrix with entries $\sum b_{rs}^g \alpha(g) t^{\phi(g)}$. Since each $\sum b_{rs}^g \alpha(g) t^{\phi(g)}$ is a $k \times k$ matrix with entries in $\mathbb{F}[t^{\pm 1}]$ we can think of $(\alpha \otimes \phi)(B)$ as a $pk \times qk$ matrix with entries in $\mathbb{F}[t^{\pm 1}]$.

Now note that

$$\det((\alpha \otimes \phi)(B_3)) = \det(\mathrm{id} - (\alpha \otimes \phi)(g_r)) = \det(\mathrm{id} - \phi(g_r)\alpha(g_r)) \neq 0$$

since $\phi(g_r) \neq 0$. Similarly $\det((\alpha \otimes \phi)(B_1)) \neq 0$ and $\det((\alpha_{\mathfrak{m}} \otimes \phi)(B_i)) \neq 0$, $i = 1, 3$ for any maximal ideal $\mathfrak{m}$. We need the following theorem which can be found in [26].

Theorem 6.2 (*[26, Theorem 2.2, Lemma 2.5, Theorem 4.7]*). *Let S be a Noetherian UFD. Let $\beta : \pi_1(M) \to \mathrm{GL}(S, k)$ be a representation and $\varphi \in H^1(M)$.*

(1) *If $\det((\beta \otimes \varphi)(B_i)) \neq 0$ for $i = 1, 2, 3$, then $H_i^{\beta}(M; S^k[t^{\pm 1}])$ is $S[t^{\pm 1}]$-torsion for all i.*

(2) *If $H_i^{\beta}(M; S^k[t^{\pm 1}])$ is $S[t^{\pm 1}]$-torsion for all i, and if $\det((\beta \otimes \varphi)(B_i)) \neq 0$ for $i = 1, 3$, then $\det((\beta \otimes \varphi)(B_2)) \neq 0$ and*

$$\prod_{i=1}^{3} \det((\beta \otimes \varphi)(B_i))^{(-1)^i} = \prod_{i=0}^{3} (\Delta_i^{\beta}(t))^{(-1)^{i+1}} = \tau(M, \varphi, \beta).$$

First assume that $\Delta_1^{\alpha_{\mathfrak{m}}}(t) \neq 0$ and

$$\|\phi\|_T = \frac{1}{k}(\deg(\Delta_1^{\alpha_{\mathfrak{m}}}(t)) - \deg(\Delta_0^{\alpha_{\mathfrak{m}}}(t)) - \deg(\Delta_2^{\alpha_{\mathfrak{m}}}(t)))$$

for all maximal ideals $\mathfrak{m}$. By Lemma 2.4 and Proposition 2.5 we get $\Delta_i^{\alpha_{\mathfrak{m}}}(t) \neq 0$ for all i, in particular $H_i^{\alpha_{\mathfrak{m}}}(M; \mathbb{F}_{\mathfrak{m}}^k[t^{\pm 1}])$ is $\mathbb{F}_{\mathfrak{m}}[t^{\pm 1}]$-torsion for all i and all maximal ideals $\mathfrak{m}$. It follows from Theorem 6.2 that $\det((\alpha_{\mathfrak{m}} \otimes \phi)(B_2)) \neq 0$. Clearly this also implies that $\det((\alpha \otimes \phi)(B_2)) \neq 0$. Since we already know that $\det((\alpha \otimes \phi)(B_i)) \neq 0$ for $i = 1, 3$ it follows from Theorem 6.2 that $H_i^{\alpha}(M; R^k[t^{\pm 1}])$ is $R[t^{\pm 1}]$-torsion for all i.

It follows from [26, Lemma 4.11] that $\Delta_0^{\alpha}(t)$ divides $\det((\alpha \otimes \phi)(B_1)) = \det(\mathrm{id} - \phi(h_s)\alpha(h_s))$ which is a monic polynomial in $R[t^{\pm 1}]$ since $\phi(h_s) \neq 0$ and since $\det(\alpha(h_s))$ is a unit. But then $\Delta_0^{\alpha}(t)$ is monic as well. The same argument (again using [26, Lemma 4.11]) shows that $\Delta_2^{\alpha}(t)$ is monic. It follows from the argument of Lemma 2.4 that $H_3^{\alpha}(M; R^k[t^{\pm 1}]) = 0$, hence $\Delta_3^{\alpha}(t) = 1$.

Denote the map $R \to R/\mathfrak{m} = \mathbb{F}_{\mathfrak{m}}$ by $\pi_{\mathfrak{m}}$. We also denote the induced map $R[t^{\pm 1}] \to \mathbb{F}_{\mathfrak{m}}[t^{\pm 1}]$ by $\pi_{\mathfrak{m}}$. It follows from Theorem 6.2 that

$$\prod_{i=0}^{3} \pi_{\mathfrak{m}}(\Delta_i^{\alpha}(t)^{(-1)^{i+1}}) = \prod_{i=1}^{3} \pi_{\mathfrak{m}}(\det((\alpha \otimes \phi)(B_i)))^{(-1)^i}$$

$$= \prod_{i=1}^{3} \det((\alpha_{\mathfrak{m}} \otimes \phi)(B_i))^{(-1)^i}$$

$$= \prod_{i=0}^{3} \Delta_i^{\alpha_{\mathfrak{m}}}(t)^{(-1)^{i+1}}$$

for all maximal ideals $\mathfrak{m}$. By assumption we get

$$\frac{1}{k}\sum_{i=0}^{3}(-1)^{i+1}\deg\left(\pi_{\mathfrak{m}}\left(\Delta_i^{\alpha}(t)\right)\right) = \frac{1}{k}\sum_{i=0}^{3}(-1)^{i+1}\deg\left(\Delta_i^{\alpha_{\mathfrak{m}}}(t)\right) = \|\phi\|_T$$

for all $\mathfrak{m}$. Since $\Delta_i^\alpha(t)$ is monic for $i = 0, 2, 3$ it follows that

$$\deg\left(\pi_{\mathfrak{m}}\left(\Delta_1^\alpha(t)\right)\right) = \deg\left(\pi_{\mathfrak{n}}\left(\Delta_1^\alpha(t)\right)\right)$$

for all maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$. Since R is a UFD it follows that $\Delta_1^\alpha(t)$ is monic. Hence $\deg\left(\pi_{\mathfrak{m}}\left(\Delta_i^\alpha(t)\right)\right) = \deg\left(\Delta_i^\alpha(t)\right)$ for all i and all maximal ideals $\mathfrak{m}$ and clearly

$$\|\phi\|_T = \frac{1}{k}\left(\deg(\Delta_1^\alpha(t)) - \deg(\Delta_0^\alpha(t)) - \deg(\Delta_2^\alpha(t))\right).$$

Now assume that $\Delta_1^\alpha(t) \in R[t^{\pm 1}]$ is monic and

$$\|\phi\|_T = \frac{1}{k}\left(\deg(\Delta_1^\alpha(t)) - \deg(\Delta_0^\alpha(t)) - \deg(\Delta_2^\alpha(t))\right).$$

The same argument as above shows that $\Delta_i^\alpha(t)$, $i = 0, 2, 3$, are monic as well. Recall that $\det(\alpha \otimes \phi)(B_i)$, $i = 1, 3$, are monic polynomials. It follows from Theorem 6.2 that

$$\det(\alpha \otimes \phi)(B_2) = \det(\alpha \otimes \phi)(B_1)\,\det(\alpha \otimes \phi)(B_3)\prod_{i=0}^{3}\left(\Delta_i^\alpha(t)\right)^{(-1)^{i+1}}$$

is a quotient of monic non-zero polynomials. In particular $\det(\alpha_{\mathfrak{m}} \otimes \phi)(B_2) = \pi_{\mathfrak{m}}(\det(\alpha \otimes \phi)(B_2)) \neq 0$. It now follows immediately from Theorem 6.2 that $H_i^{\alpha_{\mathfrak{m}}}(M; \mathbb{F}_{\mathfrak{m}}^k[t^{\pm 1}])$ is $\mathbb{F}_{\mathfrak{m}}[t^{\pm 1}]$-torsion for all i. In particular $\Delta_1^{\alpha_{\mathfrak{m}}}(t) \neq 0$. Using arguments as above we now see that

$$\begin{aligned}
\deg(\tau(M, \phi, \alpha_p)) &= \frac{1}{k}\left(\deg\left(\Delta_1^{\alpha_{\mathfrak{m}}}(t)\right) - \deg\left(\Delta_0^{\alpha_{\mathfrak{m}}}(t)\right) - \deg\left(\Delta_2^{\alpha_{\mathfrak{m}}}(t)\right)\right) \\
&= \frac{1}{k}\sum_{i=0}^{3}(-1)^{i+1}\deg\left(\Delta_i^{\alpha_{\mathfrak{m}}}(t)\right) \\
&= \frac{1}{k}\sum_{i=0}^{3}(-1)^{i+1}\deg\left(\pi_{\mathfrak{m}}\left(\Delta_i^\alpha(t)\right)\right) \\
&= \frac{1}{k}\sum_{i=0}^{3}(-1)^{i+1}\deg\left(\Delta_i^\alpha(t)\right) \\
&= \|\phi\|_T. \quad \square
\end{aligned}$$

Remark. Let $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{Z}, k)$ be a representation. Then it is in general *not* true that for a prime p we have $\Delta_1^{\alpha_p}(t) = \pi_p(\Delta_1^\alpha(t)) \in \mathbb{F}_p[t^{\pm 1}]$ (we use the notation of Proposition 6.1), not even if (M, ϕ) fibers over S^1. Indeed, let K be the trefoil knot and $\varphi : \pi_1(X(K)) \to S_3$ the unique epimorphism. Consider the representation $\alpha(\varphi) : \pi_1(X(K)) \to \mathrm{GL}(\mathbb{Z}, 2)$ as in Section 5.1. Then $\deg\left(\pi_3(\Delta_1^\alpha(t))\right) = 2$, but $\deg\left(\Delta_1^{\alpha_3}(t)\right) = 3$.

Acknowledgments

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